Response functions in phase-ordering kinetics

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We discuss the behavior of response functions in phase-ordering kinetics within the perturbation theory approach developed earlier. At zeroth order the results agree with previous gaussian theory calculations. At second order the nonequilibrium exponents λ and λ_R are changed but remain equal.

I. INTRODUCTION

The scaling properties of response functions for phase-ordering systems has been the subject of some recent [1] interest. We study these properties here using the ϕ-expansion method developed previously [2] to extend perturbation theory beyond the Gaussian level. We find, while there are no corrections generated at second-order for the exponent a governing the response function, there are second-order corrections for the nonequilibrium exponents λ and λ_R and $\lambda = \lambda_R$. We also find the associated scaling function at second order in perturbation theory.

If $\psi(\mathbf{r},t)$ is the scalar ordering field whose dynamics are driven by a time dependent Ginzburg-Landau (TDGL) model (described in detail below), then we can study the correlation function

$$C(\mathbf{r}_1 - \mathbf{r}_2, t_1, t_2) = \langle \psi(\mathbf{r}_1, t_1) \psi(\mathbf{r}_2, t_2) \rangle. \tag{1}$$

This quantity has been studied in a growth kinetics context [3] for a variety of systems and shows the scaling behavior $C(\mathbf{r}, t_1, t_2) = F(r/L(t_1), t_1/t_2)$, where $L(t_1)$ is the characteristic growth law $L(t_1) \approx t^{1/z}$ and the growth exponent is $z = 2$ for the nonconserved order parameters studied in this paper. Focus has been on the on-site correlation function $C(0, t_1, t_2) = F(0, t_1/t_2)$. For $t_1 \gg t_2$ we have $F(0, x) \approx x^{-\lambda/z}$, where λ is the well studied [4,5] nonequilibrium index.

We can also introduce an external field $B(1)$ *conjugate* to the order parameter and define the response function

$$\chi(12) = \left(\frac{\delta \langle \psi(1) \rangle}{\delta B(2)} \right)_{B=0} \tag{2}$$

evaluated at zero field. For the local ($\mathbf{r}_1 = \mathbf{r}_2$) response function it has become customary to write, $t_1 > t_2$,

$$\chi(0, t_1, t_2) = t_2^{-(1+a)} f(t_1/t_2), \tag{3}$$

where for large $x = t_1/t_2$,

$$f(x) \approx x^{-\lambda_R/z}. \tag{4}$$

The goal in the analysis is to find the exponents a and λ_R and the scaling function $f(x)$. Here we focus on the regime where both t_1 and t_2 are large enough for the system to be in the scaling regime.

Let us review some of the results found previously. The exact result [6,7] for the one-dimensional Ising model gives

$a = 0$, $f(x) \sim (x-1)^{-1/2}$, and $\lambda_R = 1$. The solution of the problem in the large n limit [8] gives the exponent

$$a = (d-2)/2, \tag{5}$$

the scaling function

$$f(x) \sim \frac{x^{d/4}}{(x-1)^{d/2}}, \tag{6}$$

and $\lambda_R = d/2$. Berthier, Barrat, and Kurchan [9] have carried out a Gaussian auxiliary function approximation calculation for this problem. This has been extended to treat one dimension in Ref. [10] with the results

$$a = (d-1)/2 \tag{7}$$

and $f(x)$ is again given by Eq. (6). Henkel, Pleimling, Godreche, and Luck in Ref. [11], using conformal invariance methods, have derived a form of the scaling function

$$f(x) \sim \frac{x^{a+1-\lambda_R/z}}{(x-1)^{(a+1)}}. \tag{8}$$

This is the same form as Eq. (6) if $\lambda_R/z = d/4$.

Numerical work [12] has focussed on smoothed integrals of the fundamental response function $\chi(t_1, t_2)$. This smoothing procedure helps with numerical sampling but can lead to qualitative differences between the fundamental response function and its smoothed counterparts for lower dimensions.

We focus here on the computation of the local response $\chi(t_1, t_2) = \chi(0, t_1, t_2)$ to second order in the ϕ expansion developed in Ref. [2] for a scalar order parameter. At zeroth order we find

$$\chi_0(t_1, t_2) = t_2^{-1-a} f_0(t_1/t_2), \tag{9}$$

with the exponent a given by Eq. (7) and

$$f_0(x) = \sqrt{\frac{2}{\pi A_0}} \frac{x^{d/4}}{[4\pi(x-1)]^{d/2}}, \tag{10}$$

where the constant A_0 is defined in Sec. IV below.

At second order in perturbation theory we again find the scaling form given by Eq. (3) with $a = \frac{1}{2}(d-1)$ but the scaling function is given now by

TABLE I. Second-order values for the exponents $\lambda = \lambda_R$ and the parameter ω from Ref. [2]).

Dimension	$\lambda = \lambda_R$	ω
1	$0.6268\ldots$	$0.4601\ldots$
2	$1.1051\ldots$	$0.6877\ldots$
3	$1.5824\ldots$	$0.9067\ldots$
Large	$d/2$	$\frac{d}{2}(\sqrt{2}-1)$

$$f(x) = \sqrt{\frac{2}{\pi A_0}} \frac{x^{\omega - 1/2}}{(4\pi)^{d/2}(x-1)^{d/2-v}}$$

$$\times [1 + \Delta_d(x) + \omega^2 2^{d-1}\bar{g}(x)], \quad (11)$$

where ω is determined by Eq. (14), v is given by Eq. (166), $\Delta_d(x)$ is given by Eqs. (140) and (136), and $\bar{g}(x)$ is given by Eqs. (157) and (163). In the large x limit, where $\Delta_d(x)$ and $\bar{g}(x)$ go to a constant, we can use Eq. (11) to identify the nonequilibrium exponent

$$\lambda_R = d + 1 - 2\omega - 2v. \quad (12)$$

Let us use the same self-consistent procedures for evaluating λ_R as used in I for λ and ω. We found in I, at this same order, that

$$\lambda = \frac{d}{2} + \omega^2 \frac{2^d M_d}{3^{d/2+1}}, \quad (13)$$

with ω given as the solution to

$$2\omega + \omega^2 2^d \left(K_d + \frac{M_d}{3^{d/2+1}} \right) = 1 + \frac{d}{2}, \quad (14)$$

$$M_d = \int_0^1 dz \frac{z^{d/2-1}}{[1+z]^d} = \frac{1}{2}\frac{\Gamma^2(d/2)}{\Gamma(d)}. \quad (15)$$

and K_d is given by Eq. (164). Solving for ω from Eq. (14) and inserting this into Eq. (13) gives the values of λ shown in Table I. Notice, however, that if we eliminate K_d from Eq. (14) using Eq. (166) for v and eliminate M_d in Eq. (14) in favor of $\lambda - d/2$ using Eq. (13), we obtain

$$2\omega + 2v + (\lambda - d/2) = 1 + d/2. \quad (16)$$

This can be rewritten as

$$\lambda = 1 + d - 2\omega - 2v. \quad (17)$$

Comparing with Eq. (12) and we find

$$\lambda = \lambda_R \quad (18)$$

and, as was found at lowest order, the nonequilibrium exponents are equal. These cancellations, coming from two very independent calculations of λ and λ_R, serve as a severe check on the validity of the algebra carried out in each.

In the following section we describe the setting up of the perturbation theory used to obtain these results.

II. INTRODUCTION OF ORDERING AUXILIARY FIELD

Consider a system where the ordering kinetics are driven by the simplest nonlinear TDGL model. Assuming that we quench a system from a disordered high temperature state to zero temperature, where the thermal noise is set to zero, we have the equation of motion

$$\frac{\partial \psi}{\partial t} = \Gamma[-V'(\psi) + c\nabla^2\psi + B] + \delta(t_1 - t_0)\psi_0, \quad (19)$$

where $V(\psi)$ is the driving potential. Typically this is chosen for simplicity to be of the ψ^4 type: $V = \frac{1}{4}(1-\psi^2)^2$. We have also included an external field $B(\mathbf{r},t)$, conjugate to the order parameter, in the equation of motion. Choosing units of time and space such that $\Gamma = c = 1$ we can write

$$\Lambda(1)\psi(1) = -V'[\psi(1)] + B(1) + \delta(t_1 - t_0)\psi_0, \quad (20)$$

where the diffusion operator

$$\Lambda(1) = \frac{\partial}{\partial t_1} - \nabla_1^2 \quad (21)$$

is introduced along with the shorthand notation that 1 denotes $(\mathbf{r}_1, t_1)$.

We show, extending the analysis in I to include an external field, that the order parameter ψ can be divided into an ordering component σ and an equilibrating component u,

$$\psi = \sigma[m] + u[m], \quad (22)$$

where $\sigma[m]$ is the solution to the Euler-Lagrange equation

$$\frac{d^2\sigma}{dm^2} \equiv \sigma_2 = V'[\sigma[m]], \quad (23)$$

connecting the degenerate states $\psi = \pm\psi_0$, $V'(\pm\psi_0) = 0$. In this equation m is taken to be the coordinate. In the case of a ψ^4 potential we have the solution

$$\sigma[m] = \tanh\left(\frac{1}{\sqrt{2}}m\right). \quad (24)$$

It is shown in Ref. [2] that the equilibrating field u decays exponentially to zero at long times. The scaling properties of the theory are carried by the ordering field $\sigma[m]$.

It was shown in I that the theory is self-consistent if the auxiliary field saisfies an equation of motion of the form

$$\Lambda(1)m(1) = \Xi(1) + B(1) + \delta(t_1 - t_0)m_0(\mathbf{r}_1), \quad (25)$$

where Ξ is a function of m which must scale as $\approx 1/L(t)$ in the scaling regime and self-consistently generates ordering. In particular, Ξ must be such that $\langle m^2(1)\rangle \approx L^2(t_1)$. In I and here we study the nonlinear model where

$$\Xi(1) = \xi(t_1)\sigma(m(1)) \quad (26)$$

and, if scaling is to hold, we find self-consistently that $\xi(t_1) \approx 1/L(t_1)$ and is independent of the field m.

III. FIELD THEORY FOR AUXILIARY FIELD

Let us consider the field theory associated with the equation of motion for $m(\mathbf{r}, t)$ given by Eqs. (25) and (26). Our development will follow the standard Martin-Siggia-Rose (MSR) [13] method in its functional integral form as developed by DeDominicis and Peliti [14]. In the MSR method the field theoretical development requires a doubling of operators to include the field M which is conjugate to m. We give here an overview of the development in I needed here to treat the response function including the coupling to an external field.

Following standard procedures, described in more detail in I, averages of interest are given as functional integrals over the fields m, M and weighted by the probability distribution $P[m, M]$:

$$\langle f(m, M) \rangle = \int \mathcal{D}m \mathcal{D}M \, P[m, M] f(m, M), \quad (27)$$

$$P[m, M] = e^{A_T(m, M)}/Z(H, h),$$

and

$$Z(h, H) = \int \mathcal{D}m \mathcal{D}M \, e^{A_T(m, M)}. \quad (28)$$

The action takes the form

$$A_T(m, M) = A(m, M) + \int d1[h(1)m(1) + H(1)M(1)], \quad (29)$$

where

$$A(m, M) = -i \int d1 \, M(1)[\Lambda(1)m(1) - \xi(1)\sigma(1) - B(1)]$$
$$- \frac{1}{2} \int d1 \int d2 \, M(1)\Pi_0(12)M(2), \quad (30)$$

with

$$\Pi_0(12) \equiv \delta(t_1 - t_0)\delta(t_1 - t_2)g(\mathbf{r}_1 - \mathbf{r}_2). \quad (31)$$

In these equations we use the notation, $\int d1 = \int dt_1 d^d r_1$, and assume that the initial field $m_0(\mathbf{r})$ is Gaussian and has a variance given by

$$\langle m_0(\mathbf{r}_1)m_0(\mathbf{r}_2) \rangle = g(\mathbf{r}_1 - \mathbf{r}_2). \quad (32)$$

We can generate correlation functions as functional derivatives in terms of sources h and H which couple to the conjugate fields.

The fundamental equations of motion are given by the identities

$$\left\langle \frac{\delta}{\delta M(1)} A_T(m, M) \right\rangle_h = 0, \quad (33)$$

$$\left\langle \frac{\delta}{\delta m(1)} A_T(m, M) \right\rangle_h = 0, \quad (34)$$

where the subscript h indicates that the average includes the source fields h and H. Taking the derivative in Eq. (33) we obtain

$$i[\Lambda(1)\langle m(1) \rangle_h - Q_1(1)] = -\int d2 \, \Pi_0(12)\langle M(2) \rangle_h + H(1) - iB(1), \quad (35)$$

where the nonlinearities are included in

$$Q_1(1) = \xi(1)\langle \sigma(1) \rangle_h, \quad (36)$$

Eq. (34) gives

$$-i[\tilde{\Lambda}(1)\langle M(1) \rangle + \hat{Q}_1(1)] = h(1),$$

where

$$\tilde{\Lambda}(1) = \frac{\partial}{\partial t_1} + \nabla_1^2$$

and

$$\hat{Q}_1(1) = \xi(1)\langle \sigma_1(m(1))M(1) \rangle. \quad (37)$$

Clearly we can go on and generate equations for all of the cumulants by taking functional derivatives of Eqs. (35) and (36). Let us introduce the notation that $G_{A_1, A_2, \ldots, A_n}(12 \cdots n)$ is the nth order cumulant for the set of fields $\{A_1, A_2, \ldots, A_n\}$, where field A_1 has argument (1), field A_2 has argument (2), et. This notation is needed when we mix cumulants with m and M. As an example

$$G_{Mmmm}(1234) = \frac{\delta^3 \langle m(4) \rangle_h}{\delta H(1)\delta h(2)\delta h(3)}. \quad (38)$$

As a shorthand for cumulants involving only m fields we write

$$G_n(12 \cdots n) = \frac{\delta^{n-1}}{\delta h(n)\delta h(n-1) \cdots \delta h(2)} \langle m(1) \rangle_h. \quad (39)$$

The equations governing the nth order cumulants are given by

$$-i[\tilde{\Lambda}(1)G_{Mm \cdots m}(12 \cdots n) + \hat{Q}_n(12 \cdots n)] = 0 \quad (40)$$

and

$$i[\Lambda(1)G_n(12 \cdots n) - Q_n(12 \cdots n)]$$
$$= -\int d\bar{1}\Pi_0(1\bar{1})G_{Mm \cdots m}(\bar{1}2 \cdots n). \quad (41)$$

The Q's are defined by

$$\hat{Q}_n(12\cdots n) = \frac{\delta^{n-1}}{\delta h(n)\,\delta h(n-1)\cdots\delta h(2)}\hat{Q}_1(1), \quad (42)$$

$$Q_n(12\cdots n) = \frac{\delta^{n-1}}{\delta h(n)\,\delta h(n-1)\cdots\delta h(2)}Q_1(1). \quad (43)$$

With this notation the equations determining the two-point functions can be written as

$$-i[\tilde{\Lambda}(1)G_{Mm}(12)+\hat{Q}_2(12)] = \delta(12), \quad (44)$$

$$i[\Lambda(1)G_2(12)-Q_2(12)] = -\int d\bar{1}\Pi_0(1\bar{1})G_{Mm}(\bar{1}2). \quad (45)$$

This is all formally exact. In order to develop perturbation theory the next step is to show that $Q_1(1)$ and $\hat{Q}_1(1)$ can be expressed in terms of the singlet probability distribution

$$P_h(x,1) = \langle\delta(x-m(1))\rangle_h. \quad (46)$$

One finds

$$Q_1(1) = \int dx\,\xi(1)\sigma(x)P_h(x,1), \quad (47)$$

$$\hat{Q}_1(1) = \int dx\,\xi(1)\sigma_1(x)\left[\langle M(1)\rangle_h + \frac{\delta}{\delta H(1)}\right]P_h(x,1). \quad (48)$$

Then any perturbation theory expansion for $P_h(x,1)$ will lead immediately to an expansion for $\hat{Q}_1(1)$ and $Q_1(1)$. We can then obtain $\hat{Q}_n$ and Q_n by functional differentiation.

The development of a perturbation theory expansion for $P_h(x,1)$ begins by using the integral representation for the δ function:

$$P_h(x,1) = \int\frac{dk}{2\pi}e^{-ikx}\Phi(k,h,1), \quad (49)$$

where

$$\Phi(k,h,1) = \langle e^{ikm(1)}\rangle_h. \quad (50)$$

The average of the exponential is precisely of the form which can be rewritten in terms of cumulants:

$$\Phi(k,h,1) = \exp\left[\sum_{s=1}^{\infty}\frac{(ik)^s}{s!}G_s(11\cdots1)\right], \quad (51)$$

where $G_s(11\cdots1)$ is the s-order cumulant for the field $m(1)$.

Consider first the lowest-order contribution to Q_n which does not vanish with the external fields h,H:

$$Q_2(12) = \int dx\,\xi(1)\sigma(x)\frac{\delta}{\delta h(2)}P_h(x,1). \quad (52)$$

We have shown that the nth order cumulants are of order $(n/2)-1$ in an expansion parameter we will develop. Expanding $\Phi(k,h,1)$ in powers of the cumulants with $n\rangle2$ and keeping terms up to the four-point cumulant, we obtain

$$P_h(x,1) = \left[1-\frac{1}{3!}G_3(111)\frac{d^3}{dx^3}+\frac{1}{4!}G_4(1111)\frac{d^4}{dx^4}+\cdots\right]\times P_h^{(0)}(x,1), \quad (53)$$

where

$$P_h^{(0)}(x,1) = \int\frac{dk}{2\pi}\Phi_0(k,h,1)e^{-ikx} \quad (54)$$

and

$$\Phi_0(k,h,1) = e^{ikG_1(1)}e^{-(1/2)k^2G_2(11)}. \quad (55)$$

Then, after taking the derivative with respect to $h(2)$, setting the external fields to zero, and neglecting all cumulants with $n\rangle2$, we obtain

$$\Phi_0(k,h=0,1) = e^{-(1/2)k^2S_2(1)} \quad (56)$$

and

$$Q_2^{(0)}(12)$$
$$= \int dx\,\xi(1)\sigma(x)\int\frac{dk}{2\pi}e^{-ikx}ikG_2(12)e^{-(1/2)k^2S_2(1)},$$

where we have defined in zero external field

$$S_2(1) \equiv G_2(11) = \langle m^2(1)\rangle. \quad (57)$$

In the scaling regime, where $S_2(1)$ is very large, we can replace $\sigma(x)\to\psi_0\mathrm{sgn}(x)$ in the integral and obtain

$$Q_2^{(0)}(12) = \xi(1)\psi_0 G_2(12)\int dx\,\mathrm{sgn}(x)$$

$$\times\int\frac{dk}{2\pi}ike^{-ikx}e^{-(1/2)k^2S_2(1)}$$

$$= \xi(1)\psi_0 G_2(12)\int dx\,\mathrm{sgn}(x)\left[-\frac{d}{dx}\Phi_0(x,1)\right].$$

In this case we integrate by parts in the integral over x and use $(d/dx)\mathrm{sgn}(x) = 2\delta(x)$ to obtain

$$Q_2^{(0)}(12) = \xi(1)\psi_0 G_2(12)2\Phi_0(0,1)$$

$$= \xi(1)\psi_0 G_2(12)\sqrt{\frac{2}{\pi S_2(1)}}. \quad (58)$$

Turning to $\hat{Q}_2(12)$ we note that it is given by taking the derivative of Eq. (48) with respect to $h(2)$. In the scaling regime where the characteristic length $L(t)$ is large we can replace $\sigma_1(x)\to\psi_0 2\delta(x)$. Then we need only consider

$$\hat{Q}_2(12) = \xi(1)\psi_0 \int dx\, 2\,\delta(x)\left[G_{Mm}(12) + \langle M(1)\rangle_h \frac{\delta}{\delta h(2)}\right.$$

$$\left. + \frac{\delta^2}{\delta h(2)\delta H(1)}\right]P_h(x,1).$$

A key observation is that as we analyze contributions to Q_n or $\hat{Q}_n$ we will find that each term consists of products of correlation functions and response functions with legs tied together by factors defined by

$$\phi_p(1) \equiv \int dx\, \text{sgn}(x) \int \frac{dk}{2\pi} ik^{2p+1}e^{-ikx}\Phi_0(k,1)$$

$$= 2\int \frac{dk}{2\pi} k^{2p} e^{-(1/2)k^2 S_2(1)}$$

$$= \left(-2\frac{d}{dS_2(1)}\right)^p \phi_0(1), \tag{59}$$

where we have used an integration by parts in going from the first to the second line and defined as

$$\phi_0(1) = 2\int \frac{dk}{2\pi} e^{-(1/2)k^2 S_2(1)} = \sqrt{\frac{2}{\pi S_2(1)}}. \tag{60}$$

Each term in the perturbation theory expansion for Q_n or $\hat{Q}_n$ will be proportional to factors of ϕ_p. The perturbation expansion is ordered by the sum of the labels p on ϕ_p. Thus a contribution with insertions $\phi_1\phi_2\phi_1$, each factor typically associated with different times, is $O(4)$. We refer to this expansion as the ϕ expansion. It should be emphasized that at this stage this is a *formal* expansion. At order n it is true that $\phi_p \approx L^{-(2p+1)}$, which is small, however it will be multiplied, depending on the quantity expanded, by positive factors of $L(t)$ such that each term in the expansion in ϕ_p has the same overall leading power with respect to $L(t)$.

To see how this expansion works let us consider first the two-point quantity $Q_2(12)$, defined by

$$Q_2(12) = \int dx\, \xi(1)\psi_0 \text{sgn}(x)\frac{\delta P_h(x,1)}{\delta h(2)}. \tag{61}$$

Using Eqs. (49) and (51) and taking the derivatives with respect to $h(2)$, it was shown in I, in the case of zero external fields, that

$$Q_2(12) = \psi_0\xi(1)\int dx\, \text{sgn}(x)\int \frac{dk}{2\pi} e^{-ikx}\Phi(k,h=0,1)$$

$$\times \sum_{s=0}^{\infty} \frac{(ik)^{2s+1}}{(2s+1)!}G_{2s+2}(11\cdots12). \tag{62}$$

Since all odd cumulants vanish in the case of zero external fields we have

$$\Phi(k,h=0,1) = \exp\left[\sum_{s=1}^{\infty} \frac{(-1)^s k^{2s}}{(2s)!}S_{2s}(1)\right], \tag{63}$$

where

$$S_{2s}(1) = G_{2s}(11\cdots1). \tag{64}$$

Let us define the set of vertices

$$\mathcal{V}_p(1) = \int dx\, \text{sgn}(x) \int \frac{dk}{2\pi} ik^{2p+1}e^{-ikx}\Phi(k,h=0,1), \tag{65}$$

which can be written, after following the same set of steps in reducing the original expression for ϕ_p, as

$$\mathcal{V}_p(1) = 2\int \frac{dk}{2\pi} k^{2p}\Phi(k,h=0,1), \tag{66}$$

which is independent of position. Then the quantity $Q_2(12)$, which appears in the equation of motion for $G_2(12)$, is given in the form

$$Q_2(12) = \psi_0\xi(1)\sum_{s=0}^{\infty} \frac{(-1)^s}{(2s+1)!}\mathcal{V}_s(1)G_{2s+2}(11\cdots12), \tag{67}$$

where we have used the definition of $\mathcal{V}_s(1)$ given by Eq. (65) in the last step.

It should be clear that the vertices $\mathcal{V}_s(1)$ are at least $O(s)$ in the ϕ expansion. By direct expansion of $\Phi(k,h=0,1)$ about $\Phi_0(k,h=0,1)$ we obtain

$$\mathcal{V}_s(1) = \phi_s(1) + \frac{S_4(1)}{4!}\phi_{s+2}(1) - \frac{S_6(1)}{6!}\phi_{s+3}(1) + \cdots. \tag{68}$$

It was found self-consistently in I that ℓth order cumulants, such as $S_\ell(1)$, are $O(\ell/2-1)$. The terms in the expansion for $\mathcal{V}_s$, given by Eq. (68), are $O(s)$, $O(s+3)$, and $O(s+5)$, respectively.

Let us turn next to $\hat{Q}_2(12)$. It was shown in I, in the same limit of zero applied field, that the nonlinear contribution to the equation of motion for G_{mM} can be written as

$$\hat{Q}_2(12) = \xi(1)\psi_0\sum_{s=0}^{\infty} \frac{(-1)^s}{(2s)!}\mathcal{V}_s(1)G^{(2s+2)}_{mm\cdots mMm}(11\cdots112). \tag{69}$$

Before going on to discuss the perturbation theory calculation of the physical response function let us made sure the theory is sensible at zeroth order where, from Eqs. (67) and (69),

$$Q_2^{(0)}(12) = \xi(1)\psi_0\phi_0(1)G_2(12) \equiv \omega_0(1)G_2(12), \tag{70}$$

$$\hat{Q}_2^{(0)}(12) = \xi(1)\psi_0\phi_0(1)G_{Mm}(12) \equiv \omega_0(1)G_{Mm}(12), \tag{71}$$

where for a scaling solution

$$\omega_0(1) = \xi(1)\psi_0\phi_0(1) \tag{72}$$

must fall off as $1/t_1$ for large t_1.

IV. GAUSSIAN THEORY

Inserting Eqs. (70) and (71) into Eqs. (44) and (45) we obtain the equations for the response function

$$-i[\tilde{\Lambda}(1)+\omega_0(1)]G_{Mm}^{(0)}(12)=\delta(12) \tag{73}$$

and the correlation function

$$i[\Lambda(1)-\omega_0(1)]G_2^{(0)}(12)=-\int d\bar{1}\Pi_0(1\bar{1})G_{Mm}^{(0)}(\bar{1}2). \tag{74}$$

It is not difficult to show that

$$i[\Lambda(1)-\omega_0(1)]G_{mM}^{(0)}(12)=\delta(12). \tag{75}$$

Using this last result we have that the correlation function can be written as

$$G_2^{(0)}(12)=\int d\bar{1}\int d\bar{2}iG_{mM}^{(0)}(1\bar{1})iG_{mM}^{(0)}(2\bar{2})\Pi_0(\bar{1}\bar{2}), \tag{76}$$

where $\Pi_0(12)$ is given by Eq. (31).

The first step in the construction of the solution to these equations is to Fourier transform Eq. (75) over space:

$$\left[\frac{\partial}{\partial t_1}+q^2-\omega_0(t_1)\right]iG_{mM}^{(0)}(q,t_1t_2)=\delta(t_1-t_2). \tag{77}$$

This first-order differential equation has the solution

$$iG_{mM}^{(0)}(q,t_1t_2)=\theta(t_1-t_2)\exp\left[\int_{t_2}^{t_1}d\tau[-q^2+\omega_0(\tau)]\right]$$

$$=\theta(t_1-t_2)R(t_1,t_2)e^{-q^2(t_1-t_2)} \tag{78}$$

and we have defined

$$R(t_1,t_2)=\exp\left[\int_{t_2}^{t_1}d\tau\omega_0(\tau)\right]. \tag{79}$$

Taking the inverse Fourier transform we obtain

$$iG_{mM}^{(0)}(r,t_1t_2)=\theta(t_1-t_2)R(t_1,t_2)\frac{e^{-r^2/4(t_1-t_2)}}{[4\pi(t_1-t_2)]^{d/2}}. \tag{80}$$

Let us turn our attention to the correlation function: Taking the Fourier transform and inserting the results for the propagators and Π_0 we obtain

$$G_2^{(0)}(q,t_1t_2)=\theta(t_1-t_0)\theta(t_2-t_0)R(t_1,t_0)$$

$$\times R(t_2,t_0)e^{-2q^2T}\tilde{g}(q), \tag{81}$$

where $\tilde{g}(q)$ is the Fourier transform of the initial correlation function and $T=(t_1+t_2)/2-t_0$. While we are primarily interested in the long-time scaling properties of our system, we can retain some control over the influence of initial condi-

tions and still be able to carry out the analysis analytically if we introduce the initial condition

$$\tilde{g}(q)=g_0e^{-(1/2)(q\ell)^2} \tag{82}$$

or

$$g(r)=g_0\frac{e^{-(1/2)(r/\ell)^2}}{(2\pi\ell^2)^{d/2}}. \tag{83}$$

Inserting this form into Eq. (81) and doing the wave number integration we obtain

$$G_2^{(0)}(\mathbf{r},t_1t_2)$$

$$=R(t_1,t_0)R(t_2,t_0)\frac{g_0}{[2\pi(\ell^2+4T)]^{d/2}}e^{-(1/2)[r^2/(\ell^2+4T)]}. \tag{84}$$

Let us turn now to the quantity $R(t_1,t_2)$ defined by Eq. (79). We assume that ω_0 has the form given by

$$\omega_0(1)=\frac{\omega}{t_1+t_c}, \tag{85}$$

where t_c is a parameter (or function of t which goes to a value t_c for large times) such that the correlations of m has a smooth early time behavior. ω is a constant we will determine. Evaluating the integral

$$\int_{t_2}^{t_1}d\tau\omega_0(\tau)=\int_{t_2}^{t_1}d\tau\frac{\omega}{t_c+\tau}=\omega\ln\left(\frac{t_1+t_c}{t_2+t_c}\right), \tag{86}$$

we obtain

$$R(t_1,t_2)=\left(\frac{t_1+t_c}{t_2+t_c}\right)^\omega. \tag{87}$$

Inserting this result back into Eq. (84) leads to the expression for the correlation function

$$G_2^{(0)}(r,t_1t_2)=g(0)\left(\frac{t_1+t_c}{t_0+t_c}\right)^\omega\left(\frac{t_2+t_c}{t_0+t_c}\right)^\omega\frac{e^{-r^2/8T}}{(8\pi T)^{d/2}}. \tag{88}$$

If we are to have a self-consistent scaling equation then the autocorrelation function $(r=0)$, at large equal times $t_1=t_2=t$, given by

$$S_2^{(0)}(t)=t^{2\omega-d/2}\frac{1}{(t_0+t_c)^{2\omega}}\frac{g_0}{(8\pi)^{d/2}}, \tag{89}$$

must have the form $S_2^{(0)}(t)=A_0t$. Comparing we see the exponent ω must be given by

$$\omega=\frac{1}{2}\left(1+\frac{d}{2}\right) \tag{90}$$

and the amplitude by

$$A_0 = \frac{1}{(t_0+t_c)^{2\omega}} \frac{g_0}{(8\pi)^{d/2}}.$$ (91)

The general expression for the correlation function can be rewritten in the convenient form

$$G_2^{(0)}(r,t_1t_2) = \sqrt{S_2^{(0)}(t_1)S_2^{(0)}(t_2)}\Phi_0(t_1t_2)e^{-(1/2)[r^2/(\ell^2+4T)]},$$ (92)

where

$$\Phi_0(t_1t_2) = \left(\frac{\sqrt{(t_1+t_c)(t_2+t_c)}}{T+t_c+t_0}\right)^{d/2}.$$ (93)

The nonequilibrium exponent is defined in the long-time limit by

$$\frac{G_2^{(0)}(0,t_1t_2)}{\sqrt{S_0(t_1)S_0(t_2)}} = \left(\frac{\sqrt{(t_1+t_c)(t_2+t_c)}}{T+t_0+t_c}\right)^{\lambda}$$ (94)

and we obtain the Ohta-Jasnow-Kawasaki (OJK) [15,3] result

$$\lambda = \frac{d}{2}.$$ (95)

Looking at equal times we have that

$$f_0(x) = \frac{G_2^{(0)}(r,tt)}{S_2^{(0)}(t)} = e^{-x^2/2},$$ (96)

where the scaled length is defined by $\mathbf{x} = \mathbf{r}/4t$. $f_0(x)$ is just the well known OJK result for the scaled auxiliary correlation function. The connection to the physical order parameter correlation function is discussed in I. We will not need these results here.

V. NONLINEAR RESPONSE

A. General expansion

We are interested in the nonlinear response function

$$\chi(12) = \frac{\delta}{\delta B(2)} \langle \sigma(1) \rangle_{h,B=0}$$

$$= i \langle \sigma(1) M(2) \rangle_{h,B=0}$$

$$= i \frac{\delta}{\delta H(2)} \langle \sigma(1) \rangle_{h,B=0},$$ (97)

where the second line follows from Eqs. (27) and (30). We have now set $B(1)=0$ and we can express the response function in terms of a functional derivative of the singlet probability distribution function just as in the computation of Q_2:

$$\chi(1,2) = i \frac{\delta}{\delta H(2)} \int dx\, \sigma(x) P_h(x,1).$$ (98)

The perturbation theory expansion for $P_h(x,1)$ was discussed in Sec. III. Using Eqs. (49) and (51) we have

$$\chi(1,2) = i \frac{\delta}{\delta H(2)} \int dx\, \sigma(x) \int \frac{dk}{2\pi} e^{-ikx} \Phi(k,h,1)$$

$$= i \int dx\sigma(x) \int \frac{dk}{2\pi} e^{-ikx} \Phi(k,h,1)$$

$$\times \sum_{s=1}^{\infty} \frac{(ik)^s}{s!} G_{s,M}(11\cdots 12).$$ (99)

We can then set the source fields to zero and obtain

$$\chi(1,2) = i \sum_{s=0}^{\infty} \frac{(-1)^s}{(2s+1)!} \mathcal{V}_s(1) G_{2s+1,M}(11\cdots 12),$$ (100)

where we introduce the same vertices $\mathcal{V}_s$ as in Sec. III. We have the explicit expansion for the physical response function to the lowest two orders:

$$\chi(1,2) = i\phi_0(1)G_{mM}(12) + i\phi_1(1)\frac{(-1)}{3!}G_{mmmM}(1112)$$

$$+ \cdots.$$ (101)

The leading term says that the physical response contains a term proportional to the MSR response function $G_{mM}(12)$.

B. Zeroth order

At zeroth order we have the response function

$$\chi_0(1,2) = i\phi_0(1)G_{mM}^{(0)}(12).$$ (102)

$\phi_0(1)$ is given by Eq. (60) and $iG_{mM}^{(0)}(12)$ is given by Eq. (80). Putting these together we obtain

$$\chi_0(1,2) = \frac{2}{\sqrt{2\pi S_2^{(0)}(1)}} \theta(t_1-t_2)R(t_1,t_2) \frac{e^{-r^2/4(t_1-t_2)}}{[4\pi(t_1-t_2)]^{d/2}}.$$ (103)

If we focus on the on-site response function

$$\chi_0(t_1,t_2) = \chi_0(0,t_1,t_2)$$

$$= \frac{2}{\sqrt{2\pi S_2^{(0)}(1)}} \theta(t_1-t_2)$$

$$\times \left(\frac{t_1+t_c}{t_2+t_c}\right)^{\omega} \frac{1}{[4\pi(t_1-t_2)]^{d/2}}.$$ (104)

If we assume $t_1 > t_2$ and write $S_2^{(0)}(1) = A_0 t$, we have

$$\chi_0(t_1,t_2) = \sqrt{\frac{2}{\pi A_0 t_1}} \left(\frac{t_1+t_c}{t_2+t_c}\right)^{\omega} \frac{1}{[4\pi(t_1-t_2)]^{d/2}}.$$ (105)

It is conventional to write this in the scaling form

$$\chi_0(t_1,t_2) = t_2^{-1-a} f_0(x),$$ (106)

where $x = t_1/t_2$. Clearly we can identify $1 + a = 1/2 + d/2$ or

$$a = \frac{d-1}{2} \tag{107}$$

and the scaling function is given by

$$f_0(x) = \sqrt{\frac{2}{\pi A_0}} \frac{x^{\omega - 1/2}}{[4\pi(x-1)]^{d/2}}, \tag{108}$$

and, using Eq. (90) for ω, we find in the large x limit the form given by Eq. (4), with

$$\lambda_R = d + 1 - 2\omega = d/2 = \lambda. \tag{109}$$

C. Second order

At second order we have two contributions. The second-order contribution to G_{mM} can be read off from Eq. (165) in I with the result

$$\chi_1^{(2)}(12) = i\phi_1(1)G_{mM}^{(2)}(12), \tag{110}$$

where

$$G_{mM}^{(2)}(12) = \int d\bar{1}d\bar{2}G_{mM}^{(0)}(1\bar{1})\Sigma^{(2)}(\bar{1}\bar{2})G_{mM}^{(0)}(\bar{2}2) \tag{111}$$

and the associated self-energy is given by

$$\Sigma^{(2)}(12) = \tfrac{1}{2}[-i\omega_1(1)]G_{mM}^{(0)}(12)[G_2^{(0)}(12)]^2[-i\omega_1(2)]. \tag{112}$$

ω_1 is given by

$$\omega_1(1) = \frac{\omega_0(1)}{S_2^{(0)}(1)}. \tag{113}$$

The second contribution at second order is given by

$$\chi_2^{(2)}(12) = i\phi_1(1)\frac{(-1)}{3!}G_{mmmM}^{(0)}(1112). \tag{114}$$

We therefore need the lowest order expression for $G_{mmmM}^{(0)}(1112)$. The lowest order expressions for all of the various four-point cumulants were worked out in I. $G_{mmmM}^{(0)}(1112)$ is given by Eq. (155) in I,

$$G_{mmmM}^{(0)}(1112) = \int d\bar{1}3G_{mM}^{(0)}(1\bar{1})[G_2^{(0)}(1\bar{1})]^2$$

$$\times[-i\omega_1(\bar{1})]G_{mM}^{(0)}(\bar{1}2). \tag{115}$$

This second-order contribution is given then by

$$\chi_2^{(2)}(12) = \int d\bar{1}\psi_0 \frac{\phi_0(1)}{S_2^{(0)}(1)}\frac{1}{2}iG_{mM}^{(0)}(1\bar{1})$$

$$\times[G_2^{(0)}(1\bar{1})]^2\omega_1(\bar{1})iG_{mM}^{(0)}(\bar{1}2). \tag{116}$$

In order to carry out the integrals contributing to $\chi^{(2)}$ it is convenient to write

$$iG_{mM}^{(0)}(12) = \theta(t_1 - t_2)R(t_1, t_2)\left(\frac{b_{12}}{2\pi}\right)^{d/2}e^{-(b_{12}/2)r_{12}^2}, \tag{117}$$

where we introduce the notation

$$b_{ij} = \frac{1}{2(t_i - t_j)} \tag{118}$$

and

$$G_2^{(0)}(12) = R(t_1, t_0)R(t_2, t_0)g_0\left(\frac{a_{12}}{2\pi}\right)^{d/2}e^{-(a_{12}/2)r_{12}^2}, \tag{119}$$

where

$$a_{ij} = \frac{1}{2(t_i + t_j)}. \tag{120}$$

Let us first evaluate $\chi_2^{(2)}(12)$ which is somewhat simpler. Inserting these forms for the correlation functions and response functions into the integral for $\chi_2^{(2)}(1,2)$ we obtain

$$\chi_2^{(2)}(1,2) = \psi_0 \frac{\phi_0(1)}{S_2^{(0)}}\frac{1}{2}\theta(t_1 - t_2)R(t_1, t_2)R^2(t_1, t_0)g_0^2 K(12), \tag{121}$$

where

$$K(12) = \int_{t_2}^{t_1}d\bar{t}_1 R^2(\bar{t}_1, t_0)\omega_1(\bar{1})$$

$$\times\left(\frac{b_{1\bar{1}}}{2\pi}\right)^{d/2}\left(\frac{a_{1\bar{1}}}{2\pi}\right)^{d}\left(\frac{b_{\bar{1}2}}{2\pi}\right)^{d/2}J(12) \tag{122}$$

and the spatial integral is given by

$$J(12) = \int d^d\bar{r}_1 e^{-([b_{1\bar{1}}+2a_{1\bar{1}}]/2)r_{1\bar{1}}^2}e^{-(b_{\bar{1}2}/2)r_{\bar{1}2}^2}. \tag{123}$$

The spatial integral can be evaluated by completing the square in the Gaussian with the result

$$J(12) = \left(\frac{2\pi}{\alpha_0}\right)^{d/2}e^{-([b_{\bar{1}2}(b_{1\bar{1}}+2a_{1\bar{1}})]/2\alpha_0)r_{12}^2}, \tag{124}$$

where

$$\alpha_0 = b_{1\bar{1}} + b_{\bar{1}2} + 2a_{1\bar{1}}. \tag{125}$$

Let us focus on the on-site correlation function ($r_{12}^2 = 0$)

$$K(t_1 t_2) = \int_{t_2}^{t_1} d\bar{t}_1 R^2(\bar{t}_1, t_0) \omega_1(\bar{1})$$

$$\times \left(\frac{b_{1\bar{1}}}{2\pi}\right)^{d/2} \left(\frac{a_{1\bar{1}}}{2\pi}\right)^d \left(\frac{b_{\bar{1}2}}{2\pi}\right)^{d/2} \left(\frac{2\pi}{b_{1\bar{1}} + b_{\bar{1}2} + 2a_{1\bar{1}}}\right)^{d/2}$$

$$= \left(\frac{1}{2\pi}\right)^{3d/2} \int_{t_2}^{t_1} d\bar{t}_1 R^2(\bar{t}_1, t_0) \omega_1(\bar{1}) W_0^{d/2}, \qquad (126)$$

where

$$W_0 = \frac{b_{1\bar{1}} a_{1\bar{1}}^2 b_{\bar{1}2}}{b_{1\bar{1}} + b_{\bar{1}2} + 2a_{1\bar{1}}}$$

$$= \{8(t_1 + \bar{t}_1)[(t_1 - t_2)(t_1 + \bar{t}_1) + 2(\bar{t}_1 - t_2)(t_1\bar{t}_1)]\}^{-1}. \qquad (127)$$

We have from the zeroth-order theory

$$S_2^{(0)}(1) = A_0 t_1 \qquad (128)$$

and

$$\omega_1(1) = \frac{\omega}{A_0 t_1^2}, \qquad (129)$$

where ω is given at zeroth order by Eq. (90). We have then

$$K(t_1 t_2) = \frac{\omega}{A_0 t_1^2} \frac{1}{8^{d/2}} \left(\frac{1}{2\pi}\right)^{3d/2} \tilde{K}(t_1, t_2), \qquad (130)$$

where

$$\tilde{K}(t_1, t_2) = \int_{t_2}^{t_1} \frac{d\bar{t}}{\bar{t}} (\bar{t}\tilde{W})^{d/2}, \qquad (131)$$

$$\tilde{W} = 8 W_0. \qquad (132)$$

Using the result

$$g_0 = A_0 t_0^{2\omega} (8\pi)^{d/2}, \qquad (133)$$

which follows from Eq. (91), and collecting coefficients, we have

$$\chi_2^{(2)}(t_1, t_2) = \psi_0 \frac{\phi_0(1) A_0}{S_2^{(0)}(t_1)} \frac{\omega}{2\pi^{d/2}}$$

$$\times \theta(t_1 - t_2) R(t_1, t_2) t_1^{2\omega} \tilde{K}(t_1, t_2). \qquad (134)$$

Looking at the time integral we change variables from $\bar{t}$ to $\bar{t} = yt_1$ and introduce $s = t_2/t_1$ so that

$$\tilde{K}(t_1, t_2) = \int_s^1 \frac{dy}{y} \left(\frac{y}{1+y}\right)^{d/2} [t_1^2(1-s)(1+y)$$

$$+ 2t_1^2(1-y)(y-s)]^{-d/2}$$

$$= t_1^{-d} \kappa(s), \qquad (135)$$

where

$$\kappa(s) = \int_s^1 \frac{dy}{y} \left(\frac{y}{1+y}\right)^{d/2} [(1-s)(1+y)$$

$$+ 2(1-y)(y-s)]^{-d/2}. \qquad (136)$$

Then the second-order correction is given by

$$\chi_2^{(2)}(t_1, t_2) = \psi_0 \phi_0(1) \frac{\omega}{2\pi^{d/2}} \theta(t_1 - t_2) R(t_1, t_2) t_1^{2\omega - 2 - d} \kappa(s)$$

$$= \psi_0 \phi_0(1) \frac{\omega}{2\pi^{d/2}} \theta(t_1 - t_2) R(t_1, t_2) t_1^{-d/2} \kappa(s). \qquad (137)$$

The zeroth-order contribution is given by

$$\chi_0(t_1, t_2) = \psi_0 \phi_0(1) \theta(t_1 - t_2) R(t_1, t_2) [4\pi(t_1 - t_2)]^{-d/2}. \qquad (138)$$

We can then combine the zeroth and $\chi_2^{(2)}$ contributions to find

$$\tilde{\chi}(t_1, t_2) = \chi_0(t_1, t_2) + \chi_2^{(2)}(t_1, t_2) = \chi_0(t_1, t_2)[1 + \Delta_d(s)], \qquad (139)$$

where

$$\Delta_d(s) = \omega 2^{d-1} (1-s)^{d/2} \kappa(s), \qquad (140)$$

and $\kappa(s)$ is given by Eq. (136). Analytically one can show $\Delta_d(0) \to \omega/2d$ as $d \to \infty$ and $\Delta_d(s) \to (\omega/2)(1-s)$ as $s \to 1$.

We turn next to the evaluation of $G_{mM}^{(2)}$. Inserting the response and correlation functions and using the properties of $R(t_1, t_2)$ we have

$$G_{mM}^{(2)}(12) = -i \frac{1}{(2\pi)^{3d/2}} \theta(t_1 - t_2) R(t_1, t_2)$$

$$\times \int_{t_2}^{t_1} d\bar{t}_1 \int_{t_2}^{\bar{t}_1} d\bar{t}_2 \frac{g_0^2}{2} \omega_1(\bar{1}) \omega_1(\bar{2})$$

$$\times (b_{1\bar{1}} b_{\bar{1}2} b_{\bar{2}2})^{d/2} a_{\bar{1}2}^d R^2(\bar{t}_1, t_0) R^2(\bar{t}_2, t_0) J_{mM}, \qquad (141)$$

where

$$J_{mM} = \int \frac{d^d \bar{r}_1 d^d \bar{r}_2}{(2\pi)^d} e^{-(b_1/2)r_{1\bar{1}}^2} e^{-(b_2/2)r_{\bar{1}\bar{2}}^2} e^{-(b_3/2)r_{\bar{2}2}^2}$$

$$= \frac{e^{-(\alpha/2)r_{11}^2}}{\mathcal{D}^{d/2}}, \tag{142}$$

where

$$b_1 = b_{1\bar{1}}, \tag{143}$$

$$b_2 = b_{\bar{1}\bar{2}} + 2a_{\bar{1}\bar{2}}, \tag{144}$$

$$b_3 = b_{\bar{2}2}, \tag{145}$$

$$\mathcal{D} = b_1 b_2 + b_2 b_3 + b_3 b_1, \tag{146}$$

and

$$\alpha = (b_1^{-1} + b_2^{-1} + b_3^{-1})^{-1}. \tag{147}$$

Again restricting ourselves to the on-site response function

$$G_{mM}^{(2)}(0, t_1, t_2) = -i \frac{1}{(2\pi)^{3d/2}} \theta(t_1 - t_2) R(t_1, t_2)$$

$$\times \int_{t_2}^{t_1} d\bar{t}_1 \int_{t_2}^{\bar{t}_1} d\bar{t}_2 \frac{g_0^2}{2} \omega_1(\bar{1}) \omega_1(\bar{2})$$

$$\tag{148}$$

$$\times (b_{1\bar{1}} b_{\bar{1}\bar{2}} b_{\bar{2}2})^{d/2} a_{\bar{1}\bar{2}}^d R^2(\bar{t}_1, t_0)$$

$$\times R^2(\bar{t}_2, t_0) \frac{1}{\mathcal{D}^{d/2}}. \tag{149}$$

After grouping the various multiplicative terms, using Eq. (133), gives

$$G_{mM}^{(2)}(0, t_1, t_2) = -i \theta(t_1 - t_2) R(t_1, t_2) \omega^2 \frac{2^{2d-1}}{(2\pi)^{d/2}} \hat{J}(t_1, t_2), \tag{150}$$

where

$$\hat{J}(t_1, t_2) = \int_{t_2}^{t_1} d\bar{t}_1 \int_{t_2}^{\bar{t}_1} d\bar{t}_2 (\bar{t}_1 \bar{t}_2)^{2\omega - 2} (W)^{d/2}, \tag{151}$$

with

$$W = \frac{b_{\bar{1}\bar{2}} a_{\bar{1}\bar{2}}^2}{1 + b_2 b_3^{-1} + b_2 b_1^{-1}}. \tag{152}$$

After some algebra we find

$$W = \frac{1}{8(\bar{t}_1 + \bar{t}_2)} \frac{1}{[(t_1 - t_2)(3\bar{t}_1 - \bar{t}_2) - 2(\bar{t}_1 - \bar{t}_2)^2]}. \tag{153}$$

After inserting this result for W back into Eq. (151), using $\omega = d/2 + 1$, which is valid at this order, and letting $\bar{t}_1 = (t_1 - t_2)y_1$ and $\bar{t}_2 = (t_1 - t_2)y_2$ we obtain

$$\hat{J}(t_1, t_2) = \frac{1}{[8(t_1 - t_2)]^{d/2}} g(\tau), \tag{154}$$

where

$$g(\tau) = \int_\tau^{1+\tau} \frac{dy_1}{y_1} \int_\tau^{y_1} \frac{dy_2}{y_2} \left(\frac{y_1 y_2}{y_1 + y_2} \right)^{d/2}$$

$$\times \frac{1}{[3y_1 - y_2 - 2(y_1 - y_2)^2]^{d/2}} \tag{155}$$

and

$$\tau = \frac{t_2}{t_1 - t_2} = \frac{1}{t_1/t_2 - 1}. \tag{156}$$

A more useful form for $g(\tau)$ is obtained if we let $y_2 = y_1 z$ to obtain

$$g(\tau) = \int_\tau^{1+\tau} \frac{dy_1}{y_1} \int_{\tau/y_1}^1 dz f(z, y_1), \tag{157}$$

where

$$f(z, y_1) = \frac{z^{d/2-1}}{[(1+z)[(3-z) - 2y_1(1-z)^2]^{d/2}}. \tag{158}$$

This means that the MSR on-site response function can be written up to second order

$$G_{mM}(0, t_1, t_2) = -i \theta(t_1 - t_2) \frac{R(t_1, t_2)}{[4\pi(t_1 - t_2)]^{d/2}}$$

$$\times [1 + \omega^2 2^{d-1} g(\tau)]. \tag{159}$$

Pulling together all of the results for the physical response function we have up to second order

$$\chi(t_1, t_2) = t_2^{-1-a} f(t_1/t_2), \tag{160}$$

with

$$a = \tfrac{1}{2}(d - 1) \tag{161}$$

and

$$f(x) = \sqrt{\frac{2}{\pi A_0}} \frac{x^{\omega - 1/2}}{[4\pi(x - 1)]^{d/2}} [1 + \Delta_d(s) + \omega^2 2^{d-1} g(\tau)], \tag{162}$$

where $x = t_1/t_2$. It is not difficult to show that in the small τ limit

$$g(\tau) = -K_d \ln \tau + \bar{g}(\tau), \tag{163}$$

where

$$K_d = \int_0^1 dz \, \frac{z^{d/2-1}}{[(1+z)(3-z)]^{d/2}}. \quad (164)$$

K_d also appeared in the analysis in I. If our perturbation theory results are to make sense we must exponentiate the $-K_d \ln \tau$ contribution

$$1 - v \ln \tau \approx \tau^{-v} = (x-1)^v, \quad (165)$$

where in the last term we have used Eq. (156) and

$$v = \omega^2 2^{d-1} K_d. \quad (166)$$

Putting this result back into Eq. (162) we obtain our final result for the scaling function given by Eq. (11).

VI. CONCLUSIONS

We have used perturbation theory to explore the nature of the scaling solutions for the local scalar order parameter response function. At lowest, Gaussian level, order we find agreement with previous work. Going to the next order we find that the scaling index a, defined by Eq. (3), is unchanged from its Gaussian level value of $a = (1/2)(d-1)$. The non-equilibrium exponents λ and λ_R are found to be equal at lowest and second order in perturbation theory with explicit values given as a function of d. Expressions for the scaling function are also available.

It is not at all clear if there is a general proof that $\lambda = \lambda_R$. The validity of this result at second order in perturbation theory is very suggestive. It will also be interesting to see whether, within this perturbation theory, this result holds for the case [16] of vector order parameters ($n > 1$) where the analysis is more involved.

[1] A brief review is given later in this section.

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